On a Spatial Smoothing Technique for Multiple Source Location

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Abstract—**This correspondence presents an alternate proof for the spatial smoothing technique recently proposed for multiple source location. Specifically, it is proved that the number of subarrays required to ensure a nonsingular covariance matrix is $m + 1$ where m is the nullity of the source covariance matrix. Also, it is shown that this number can be reduced if a specific basis can be found for the null eigenspace of the covariance matrix satisfying certain properties.**

One of the main impediments in successfully applying eigenspace techniques to multiple source location has been the possibility that the underlying source covariance matrix can be singular. Consequently, the eigenvectors corresponding to the minimal eigenvalue of the received covariance matrix may not all be orthogonal to the direction vectors of the sources [1]–[3]. Until recently, this situation could not be handled in a satisfactory manner. The spatial smoothing technique of Evans *et al.* [4] is an elegant solution and the breakthrough required to solve the multiple source location problem.

There have been several proofs of this technique notably by Shan *et al.* [5]. In the following, we present an alternate proof that relies on the properties of eigenvectors. More importantly, we show that the number of subarrays required for spatial smoothing is $m + 1$ where m is the nullity of the source covariance matrix. We also show that this number can be further reduced if the null eigenspace of the source covariance matrix possesses a basis with special properties. This is of interest to the special case of the block diagonal covariance matrices discussed by Shan *et al.*

We follow the notation of [5] to simplify our presentation. Let S be the covariance matrix and $D = \text{diag}\ \{\exp(-j\omega_0\tau_1), \exp(-j\omega_0\tau_2), \cdots, \exp(-j\omega_0\tau_q)$ where $\tau_i = (d/c)\sin\theta_i$ is the phase factor for source i. Also let

$$\bar{S} = \sum_{k=1}^{M} D^{k-1} S (D^\dagger)^{k-1}.$$

For convenience let $E = D^\dagger$. We now prove $\bar{S}$ is of full rank when $M = m + 1$ where m is the nullity of S. Partition the eigenvectors of S into two sets $\{x_i\}$ and $\{y_i\}$ such that $\{x_i\}$ have positive (nonzero) eigenvalues and $\{y_i\}$ have zero eigenvalues. We show that $\bar{S}$ is of full rank by demonstrating $\bar{S}x_i \neq 0$ and $\bar{S}y_i \neq 0$, or equivalently, $x_i^\dagger \bar{S} x_i > 0$ and $y_i^\dagger \bar{S} y_i > 0$. $\bar{S}$ is the sum of semipositive definite matrices, and consequently,

$$x_i^\dagger S x_i > 0 \Rightarrow x_i^\dagger \bar{S} x_i > 0.$$

The following simple lemma is needed to prove $y_i^\dagger \bar{S} y_i > 0$.

Lemma: The null unit eigenvectors of the covariance matrix S cannot be the unit Cartesian vectors $(1, 0, \cdots, 0)$, $(0, 1, \cdots, 0)$, $\cdots$, and $(0, 0, \cdots, 1)$.

The proof follows from the fact that the diagonal elements of S must be positive (and nonzero).

Note:

$$y_i^\dagger \bar{S} y_i = \sum_{k=1}^{m+1} y_i D^{k-1} S E^{k-1} y_i = \sum_{l=0}^{m} z_{i,l}^\dagger S z_{i,l}$$

where $z_{i,l} = E^l y_i$. Assume y_i has w_i of its elements nonzero. Consider the following cases.

Case 1: $q \geq w_i > m$. For this case it is easy to see vectors $\{z_{i,l}\}$, $l = 0 \cdots m$ are all independent. If they are not, we have

$$\sum_{l=0}^{m} a_l z_{i,l} = \sum_{l=0}^{m} a_l E^l y_i = 0 \quad \text{for some } \{a_l\}.$$

This in turn implies $\Sigma_{l=0}^{m} a_l E^l$ has a minimum of $w_i > m$ zero diagonal elements, i.e., there exists a polynomial $\Sigma_{l=0}^{m} a_l x^l$ with a number of roots greater than m, an impossibility. Thus, vectors $z_{i,l}$, $l = 0 \cdots m$ cannot all belong to the m-dimensional null space of S. Hence, $y_i^\dagger \bar{S} y_i > 0$.

Case 2: $m \geq w_i \geq 1$. In this case, $z_{i,l}$, $l = 0 \cdots w_i - 1$ are all independent which can be proved by using a polynomial argument similar to the one used in case 1. They all have their w_i nonzero elements in the same places. If all of these vectors do not lie in the null space of S, then $y_i^\dagger \bar{S} y_i > 0$. If they do, they constitute a set of basis vectors for w_i-dimensional subspace of the null space of S. From these basis vectors, one can obviously construct Cartesian unit vectors that lie in this null space, a contradiction of the lemma.

It is interesting to observe that the situation in case 2 can be generalized to obtain the following result. If a basis of vectors is found for the null space of S such that any vector in that basis contains at the most w elements, then the number of subarrays necessary can be reduced from $m + 1$ to w. This is the situation Shan *et al.* [5] consider in their special case where the source covariance matrix is block diagonal.

Thus, to resolve q sources whose covariance matrix has a nullity of m (and rank of $r = q - m$), we need at least a linear array containing $m + 1$ sets of $q + 1$ contiguous elements, i.e., a linear array containing $q + m + 1 = 2q - r + 1$ elements. Incidentally, this is the same bound given by Di [6] using a different matrix decomposition technique.

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Spatial Smoothing and a Class of Stochastic Filters

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Abstract—**This correspondence presents a class of stochastic filters that realize spatial smoothing, a concept recently proposed to counter the effects of rank degeneracy in multiple source location and adaptive signal processing. It is shown that these filters are economic computationally, and thus preferable to direct computation of covariance matrices for achieving spatial smoothing.**

Recently, the concept of spatial smoothing has been shown to solve the problem of rank degeneracy in multiple source location and adaptive filtering [1], [2]. The concept appears to have been developed independently by Evans *et al.* [3] and Henderson [4]. (It is certainly possible that other researchers might have come across a similar concept.) We will describe a stochastic filter, referred to as a democratic filter (DF), for realizing spatial smoothing without explicit smoothing of computed covariance matrices. By using this filter, only one covariance matrix needs to be computed and, as a consequence, computational load for performing spatial smoothing is reduced. The choice of the name for the filter will be explained at the end of the correspondence.

Let $X(1, n) = [x_1, x_2, \cdots, x_n]^T$ be a column vector of n observations (in multiple source location, they correspond to the outputs of the array elements, and in adaptive filtering, they correspond to the outputs at the taps of the transversal filter). We follow the notation $X(i, p)$ to indicate column vectors starting with x and having p elements, i.e., $X(i, p) = [x_i, x_{i+1}, \cdots, x_{i+p-1}]$. Define a democratic filter $\mathrm{DF}(n, m)$ as follows:

$$\mathrm{DF}(n, m) : X(1, n) \rightarrow Y(1, m) = [y_1, y_2, \cdots, y_m]$$

$$a_1 X(1, m) + a_2 X(2, m) + \cdots + a_{n-m+1} X(n - m + 1, m) = Y(1, m) \tag{1}$$

where $a_1, a_2, \cdots, a_{n-m+1}$ are chosen to be real independent variables with the property

$$E\{a_i a_j\} = \delta_{i,j}.$$

Note (1) can be rewritten as

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_{n-m+1}x_{n-m+1} &= y_1 \\ a_1x_2 + a_2x_3 + \cdots + a_{n-m+1}x_{n-m+2} &= y_2 \\ \cdots \\ a_1x_m + a_2x_{m+1} + \cdots + a_{n-m+1}x_n &= y_m. \end{aligned} \tag{2}$$

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The covariance matrix R_y of $Y(1, m)$ is simply the sum of R_1, R_2, $R_3, \cdots, R_{n-m+1}$ where R_i is the covariance matrix of $X(i, m)$. Thus, the output of the filter has a covariance matrix which is the sum of $(n - m + 1)$ covariance matrices of dimension $m \times m$. To compute the smoothed covariance matrix by using the filter, we require approximately $m(n - m)$ multiplications for the filter, and $m^2/2$ multiplications for the covariance matrix. If the smoothed covariance matrix has to be computed explicitly, we require $m^2(n - m)/2$ multiplications. Thus, for $m = n/2$, there will be $3n^2/8$ multiplications for the democratic filter, and $n^3/8$ for direct computation.

For multiple source location, choose $2n - 1$ elements in the linear array [5], [6] and select a democratic filter $\mathrm{DF}(2n - 1, n)$ to generate an $n \times n$ covariance matrix. This matrix will have its noise subspace eigenvectors totally orthogonal to the signal subspace eigenvectors, and it will be possible to determine the angular locations of up to $n - 1$ sources regardless of whether the sources are completely correlated or not. For adaptive processing, a similar approach can be used to alleviate the effects of signal source correlation.

The reason we call the proposed filter a democratic filter is its ability to generate an output that contains the strongly correlative properties of the input signal. (This can be seen from the "covariance matrix summing" property of the filter; strongly correlated components tend to be enhanced and weakly or noncorrelated components will disappear.) As the elements of the filter behave in an independent, random manner, it is surprising that it produces an output that reflects the collective (correlative) strength of the input signal. This is remniscent of a democratic society where the people operate in a free, independent manner and yet the society reflects the collective strength of its people.

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Corrections and Modifications to "On a Spatial Smoothing Technique for Multiple Source Location"

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In the above correspondence[1] it is shown that the number of subarrays required for spatial smoothing is $m + 1$, where m is the nullity of the source covariance matrix, and that this number can be reduced for some special cases. Although the main results are valid, the proofs are flawed due to a wrong starting premise. We present the correct proof for the main result of the correspondence, and restate conditions under which the number of subarrays can be reduced.

The starting premise that if $x_i^\dagger \bar{S} x_i > 0$ and $y_i^\dagger \bar{S} y_i > 0$, then $\bar{S}$ is positive definite and nonsingular does not hold. It is true only when $\{x_i\}$ and $\{y_i\}$ are orthogonal to each other with respect to $\bar{S}$. For the correct proof that $\bar{S}$ is nonsingular, we have to prove that $z_i^\dagger \bar{S} z_i > 0$ for any z_i. The proof is exactly the same as given for y_i in the correspondence.[1]

To consider the situation where the number of subarrays can be reduced, let $\{z_i\}$ be a set of orthogonal eigenvectors for $\bar{S}$. If the number of nonzero elements for any eigenvector in this set is w, then the number of subarrays can be reduced to w. The proof follows from the fact that $\{z_i\}$ are orthogonal to each other with respect to $\bar{S}$ and, hence, $z_i^\dagger \bar{S} z_i > 0$ for all i implies $v_i^\dagger \bar{S} v_i > 0$ for any v_i. By applying the proof for case 2 to z_i, we obtain the desired result. When the source covariance matrix is block diagonal, $\bar{S}$ is also block diagonal and, hence, $\{z_i\}$ can be selected so that it has at the most w elements, where w is the size of the largest diagonal block.

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