Constrained least-squares image restoration: an improved computational scheme

S. S. Reddi

In this short paper we present an improved computational technique to solve the Fredholm integral equation of the first kind which arises in image restoration and other inverse filtering applications. The technique is based on the works of Phillips, Twomey, and Hunt. We show that when the integral represents a convolution, the integral equation can be solved iteratively with each iteration requiring $O(n)$ operations, where n is the number of sample points or observations. When there are p iterations to find the final solution to the integral equation, the present technique is approximately $(1 + p/4)$ times faster than the implementation suggested by Hunt. In image processing application areas where p has been observed to be between 3 and 12, the technique can reduce computation by a factor of 2 to 4.

Introduction

Hunt[1] in his paper demonstrated that application of constrained least squares estimation to image restoration is computationally feasible by use of fast Fourier transform techniques. The basic integral equation of image restoration,

$$g(x,y) = \int_0^y \int_0^x h(x - x_1, y - y_1) f(x_1,y_1) dx_1 dy_1 + \epsilon(x,y)$$

is solved for $f(x,y)$ by applying the techniques of Phillips[2] and Twomey.[3] In this paper we present a technique which has a better computational bound and achieves a speed up by a factor of 1.75 to 4 in practice over Hunt's approach.

Proposed Method

We consider only the following 1-D image restoration equation:

$$g(x) = \int_0^x h(x - x_1) f(x_1) dx_1 + \epsilon(x), \tag{1}$$

since extension to two dimensions is direct. The integral Eq. (1) is discretized and represented as

$$\{g\} = \{h\} \boxtimes \{f\} + \{\epsilon\}. \tag{2}$$

Here $\{x\}$ represents the sequence $\{x_0, x_1, \ldots, x_{n-1}\}$, and $\boxtimes$ indicates sequence convolution operation (see Bracewell[4] for a discussion of this operation). Let $|\{x\}|^2 = x_0^2 + x_1^2 + \ldots + x_{n-1}^2$. We solve for $\{f\}$ in Eq. (2) by finding an estimate $\{\hat{f}\}$ which satisfies the following criterion:

$$\begin{aligned} &\text{Minimize } |\{\hat{f}\} \boxtimes \{c\}|^2 \\ &\text{subject to } |\{g\} - \{h\} \boxtimes \{\hat{f}\}|^2 = |\{\epsilon\}|^2 = e. \end{aligned} \tag{3}$$

Here $\{c\}$ is usually selected to be $\{1, -2, 1\}$ so that the resulting solution is smooth and noise-free.

Define finite Fourier transforms[5] as in the following:

$$X_p = \sum_{j=0}^{n-1} x_p W^{-pj} \qquad x_p = \frac{1}{n} \sum_{j=0}^{n-1} X_p W^{pj},$$

where $W = \exp(i2\pi/n)$. Let $\{\hat{F}\}$, $\{C\}$, $\{H\}$, $\{G\}$, and $\{E\}$ be the respective Fourier transforms of $\{\hat{f}\}$, $\{c\}$, $\{h\}$, $\{g\}$, and $\{\epsilon\}$. Also let $\{XY\} = \{X_0Y_0, X_1Y_1, \ldots, X_{n-1}Y_{n-1}\}$ represent multiplication of two sequences $\{X\}$ and $\{Y\}$. Then it follows that

$$\begin{aligned} &\text{Minimize} \quad |\{\hat{F}C\}|^2 \\ &\text{subject to} \quad |\{G\} - \{H\hat{F}\}|^2 = |\{E\}|^2 \end{aligned} \tag{4}$$

is equivalent to Eq. (3) because of convolution and Parseval's theorems.[5] Hence if $\hat{f}$ satisfies Eq. (3), its Fourier transform $\hat{F}$ satisfies Eq. (4) and vice-versa.

To recast Eq. (4) in matrix form we use the following notation. Let $[F]$ represent the column matrix:

$$[F] = \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{n-1} \end{bmatrix}$$

and $\langle F \rangle$ the diagonal matrix:

When this work was done, the author was with W. W. Gaertner Research, Inc., Stamford, Connecticut 06903; he is now with Hughes Aircraft Company, Fullerton, California 92634.

Received 9 November 1977.
0003-6935/78/0801-2340$0.50/0.

$$\langle F\rangle = \begin{bmatrix} F_0 & & \\ & F_1 \square & \\ \square & \ddots & \\ & & F_{n-1} \end{bmatrix},$$

where $\{F\} = \{F_0, F_1, \ldots, F_{n-1}\}$. Then Eq. (4) can be expressed in matrix form as

$$\begin{aligned} &\text{Minimize } [\hat{F}]^{\dagger}\langle C\rangle^{\dagger}\langle C\rangle[\hat{F}] \\ &\text{subject to } ([G] - \langle H\rangle[\hat{F}])^{\dagger}([G] - \langle H\rangle[\hat{F}]) \\ &\quad = [E]^{\dagger}[E] = ne, \end{aligned} \tag{5}$$

where † represents the complex conjugate-transpose operation.

The Appendix presents a proof that $[\hat{F}]$ which satisfies Eq. (5) can be given as

$$[\hat{F}] = \frac{\langle H\rangle^{\dagger}}{\langle H\rangle^{\dagger}\langle H\rangle + \gamma\langle C\rangle^{\dagger}\langle C\rangle}[G], \tag{6}$$

where γ is such that the equality constraint in Eq. (5) holds. Since $\langle H\rangle$ and $\langle C\rangle$ are diagonal we get

$$[F] = \begin{bmatrix} H_0^*/(|H_0|^2 + \gamma|C_0|^2) & & \square \\ & H_1^*/(|H_1|^2 + \gamma|C_1|^2) & \\ \square & \ddots & \\ & & H_{n-1}^*/(|H_{n-1}|^2 + \gamma|C_{n-1}|^2) \end{bmatrix} \begin{bmatrix} G_0 \\ G_1 \\ \vdots \\ G_{n-1} \end{bmatrix}.$$

Therefore the equality constraint of Eq. (5) can be rewritten as

$$\sum_{j=0}^{n-1} |G_j|^2[1 - |H_j|^2/(|H_j|^2 + \gamma|C_j|^2)]^2 = ne \tag{7}$$

after some simple algebraic manipulations. Letting $\alpha_j = |G_j|^2$, $\beta_j = |H_j|^2/|C_j|^2$, $\lambda = 1/\gamma$, and $\delta = ne$, Eq. (7) becomes

$$\sum_j \frac{\alpha_j}{(1 + \beta_j\lambda)^2} = \delta, \tag{8}$$

where j is such that $C_j \neq 0$.

We can determine λ iteratively from the nonlinear Eq. (8), and once λ is determined $[\hat{F}]$ can be determined from Eq. (6) and hence $\{\hat{f}\}$. It can be noted that the left-hand side of Eq. (8) is monotonically decreasing and

$$\sum_{j=0}^{n-1} \alpha_j$$

is greater than δ. Thus a unique positive value for λ exists which satisfies Eq. (8).

If it takes p iterations to find the value of λ (or γ), it can be seen that Hunt's method requires $(4 + p)$ fast Fourier transforms, whereas the present method requires only four fast Fourier transforms. Assuming that each iteration to solve the nonlinear Eq. (8) takes $O(n)$ operations, we see that the proposed method can speed up the computation by approximately $(1 + p/4)$ times [since the complexity of a Fourier transform is $o(n \log n)$]. For 2-D image processing with $N \times N$ picture elements a detailed computational analysis reveals that Hunt's algorithm requires $M_H = 16N^2 \log_2 N + 4N^2 + p_1(4N^2 \log_2 N + 3N^2)$ multiplications and $D_H = 3N^2/2 + p_1 N^2$ divisions, and the present approach requires $M_p = 16N^2 \log_2 N + 5N^2 + p_2 N^2$ multiplications and $D_p = 2.5N^2 + p_2 N^2$ divisions.

Here p_1 and p_2 are the iterations required to find the final $\{\hat{f}\}$ in Hunt's method and the final λ in the present approach, respectively. Assuming that Newton-Raphson's scheme is used in both the approaches we have $p_1 = p_2 = p$. Hunt reports that p is observed to be between 3 and 12, and thus for a typical value of $p = 8$ and $N = 512$ we have

$M_H = 120.6 \times 10^6$ multiplications,
$D_H = 2.5 \times 10^6$ divisions,
$M_P = 41.2 \times 10^6$ multiplications,
$D_P = 10^6$ divisions.

Thus for this case the speed up is nearly three times.

Appendix

In this Appendix we show that the optimal solution f which minimizes $f^{\dagger}Cf$ subject to

$$(g - Hf)^{\dagger}(g - Hf) - k = 0$$

is given by

$$f = (H^{\dagger}g)/(H^{\dagger}H + \gamma C),$$

where γ is a constant. Here f and g are $n \times 1$ complex column matrices; and $H = \mathrm{diag}(H_0, H_1, \ldots, H_{n-1})$ and $C = \mathrm{diag}(C_0, C_1, \ldots, C_{-1})$ are diagonal complex and real matrices, respectively. Let the jth elements of f, g, and H be $f_{j1} + if_{j2}$, $g_{j1} + ig_{j2}$, and $H_{j1} + iH_{j2}$, respectively. Form the Lagrangian

$$\Phi = f^{\dagger}Cf + \lambda[(g - Hf)^{\dagger}(g - Hf) - k]$$

and set $\partial\Phi/\partial f_{j1} = \partial\Phi/\partial f_{j2} = 0$ to find the optimal solution. Since

$$f^{\dagger}Cf = \sum_{j=0}^{n-1} C_j(f_{j1}^2 + f_{j2}^2),$$

$$\partial(f^{\dagger}Cf)/\partial f_{j1} = 2C_j f_{j1} \tag{A1}$$

$$\partial(f^{\dagger}Cf)/\partial f_{j2} = 2C_j f_{j2}. \tag{A2}$$

Let

$$W = (g - Hf)^{\dagger}(g - Hf) = \sum_{j=0}^{n-1} (g_j - H_j f_j)^*(g_j - H_j f_j).$$

Then

$$\begin{aligned} \partial W/\partial f_{j1} &= (g_j - H_j f_j)^*(-H_j) + (-H_j)^*(g_j - H_j f_j) \\ &= -2(g_{j1}H_{j1} + g_{j2}H_{j2}) + 2|H_j|^2 f_{j1}. \end{aligned} \tag{A3}$$

Similarly

$$\partial W/\partial f_{j2} = -2(g_{j2}H_{j1} - g_{j1}H_{j2}) + 2|H_j|^2 f_{j2}. \tag{A4}$$

Combining Eqs. (A1) and (A3) and noting $\partial\Phi/\partial_{j1} = 0$,

$$C_j f_{j1} - \lambda[(g_{j1}H_{j1} + g_{j2}H_{j2}) - |H_j|^2 f_{j1}] = 0.$$

Similarly $C_j f_{j2} - \lambda[(g_{j2}H_{j1} - g_{j1}H_{j2}) - |H_j|^2 f_{j2}] = 0$. Hence $C_j f_j - \lambda(H_j^* g_j - |H_j|^2 f_j) = 0$ for an optimal solution. Therefore $f = (H^{\dagger}g)/(H^{\dagger}H + \gamma C)$, where $\gamma = 1/\lambda$.

References

1. B. R. Hunt, IEEE Trans. Comput. **C-22,** 805 (1973).
2. D. L. Phillips, J. Assoc. Comput. Mach. **9,** 84 (1962).
3. S. Twomey, J. Assoc. Comput. Mach. **10,** 97 (1963).
4. R. Bracewell, *The Fourier Transform and its Applications* (McGraw-Hill, New York, 1965).
5. J. W. Cooley, P. A. Lewis, and P. D. Welch, IEEE Trans. Audio Electroacoust. **AU-17,** 77 (1969).