2

Optimal Two- and Three-Stage Production Schedules With Set-up Time Included

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1. TWO-STAGE PRODUCTION SCHEDULE

Let us consider a typical multistage problem formulated in the following terms by R. Bellman:

"There are n items which must go through one production stage or machine and then a second one. There is only one machine for each stage. At most one item can be on a machine at a given time.

"Consider $2n$ constants A_i, B_i, $i = 1, 2, \ldots, n$. These are positive but otherwise arbitrary. Let A_i be the set-up time plus work time of the ith item on the first machine, and B_i be the corresponding time on the second machine. We seek the optimal scheduling of items in order to minimize the total elapsed time."

A simple decision rule leads to an optimal scheduling of the items minimizing the total elapsed time for the entire operation. For

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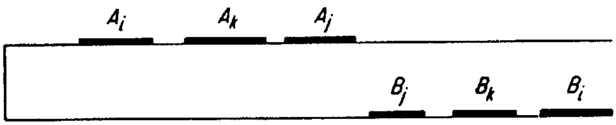
example, the decision rule permits one optimally to arrange twenty production items in about five minutes by visual inspection.

In the second section a three-stage problem is also discussed and solved for a restricted case.

Lemma 1. The production sequence on either machine can be made the same as that of the other machine without loss of time.

Proof. On the time scales for each machine place the A's and B's in any position *subject to the rules*; i.e., the start of a B_j must be to the right of the end of an A_j. If the orders are different, the elements out of order will be placed something like the following.

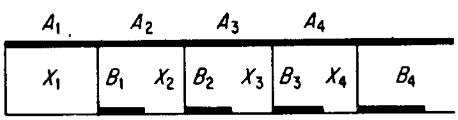


Then without loss of time we can make the ordering of stage 1 the same as the ordering of stage 2 by successive interchanges, starting from the left of consecutive pairs of those items which are out of order. Symmetrically we could order items on stage 2 to match that of stage 1.

Next, since the orders are now the same, we may start each item as soon as possible to minimize the total time. Thus there are no delay times on the first stage.

Notation. Let X_i be the inactive period of time for the second machine immediately before the ith item comes onto the second machine.

If, for example, we consider the sequence $S = 1, 2, 3, \ldots, n$, we have the following time scales for each machine:



We have

$$X_1 = A_1$$

$$X_2 = \max(A_1 + A_2 - B_1 - X_1, 0)$$

$$X_1 + X_2 = \max(A_1 + A_2 - B_1, A_1)$$

$$X_3 = \max\left(\sum_1^3 A_i - \sum_1^2 B_i - \sum_1^2 X_i, 0\right)$$

$$\sum_1^3 X_i = \max\left(\sum_1^3 A_i - \sum_1^2 B_i, \sum_1^2 X_i\right)$$

$$= \max\left(\sum_1^3 A_i - \sum_1^2 B_i, \sum_1^2 A_i - B_1, A_1\right).$$

In general,

$$\sum_1^n X_i = \max_{1 \leq u \leq n} K_u$$

where

$$K_u = \sum_{i=1}^{u} A_i - \sum_{i=1}^{u-1} B_i.$$

Let

$$F(S) = \max_{1 \leq u \leq n} K_u.$$

We want a sequence S^* such that $F(S^*) \leq F(S_0)$ for any S_0.

Solution of Problem. Consider S' the sequence formed by interchanging the jth and the $j+1$-st items in S; then

$$F(S') = \max_{1 \leq u \leq n} K'_u$$

where

$$K'_u = \sum_{i=1}^{u} A'_i - \sum_{i=1}^{u-1} B'_i$$

and

$$A'_i = A_i, \qquad B'_i = B_i \qquad \text{for } i \neq j,\ j+1$$

$$A'_j = A_{j+1}, \qquad B'_j = B_{j+1}, \qquad A'_{j+1} = A_j, \qquad B'_{j+1} = B_j.$$

Then

$$K'_u = K_u \qquad \text{if } u \neq j,\ j+1.$$

Thus $F(S') = F(S)$ unless possibly if $\max(K_j,\ K_{j+1}) \max \neq (K'_j,\ K'_{j+1})$.

Theorem 1. An optimal ordering is given by the following rule. Item (j) precedes item $(j+1)$ if

$$\max(K_j, K_{j+1}) < \max(K'_j, K'_{j+1}). \tag{1}$$

If there is equality, either order is optimal, provided it is consistent with all the definite preferences (see Case 4, Lemma 2).

By subtracting $\sum_1^{j+1} A_i - \sum_1^{j-1} B_i$ from each term in Relation (1), it becomes

$$\max(-B_j, -A_{j+1}) < \max(-B_{j+1}, -A_j)$$

or

$$\min(A_j, B_{j+1}) < \min(A_{j+1}, B_j). \tag{2}$$

This ordering is transitive (proof follows), thus leading to a sequence S^*, unique except for some indifferent elements.

Then $F(S^*) \leq F(S_0)$ for any sequence S_0, since S^* can be obtained from S_0 by successive interchanges of consecutive items, according to Relation (2), and each interchange will give a value of F smaller than or the same as before.

Lemma 2. Relation (2) is transitive. Suppose $\min(A_1, B_2) \leq \min(A_2, B_1)$ and $\min(A_2, B_3) \leq \min(A_3, B_2)$. Then

$$\min(A_1, B_3) \leq \min(A_3, B_1)$$

except possibly when item 2 is indifferent to both items 1 and 3.

Proof.

Case 1. $A_1 \leq B_2, A_2, B_1$ and $A_2 \leq B_3, A_3, B_2$.
Then $A_1 \leq A_2 \leq A_3$ and $A_1 \leq B_1$ so that $A_1 \leq \min(A_3, B_1)$.

Case 2. $B_2 \leq A_1, A_2, B_1$ and $B_3 \leq A_2, A_3, B_2$.
Then $B_3 \leq B_2 \leq B_1$ and $B_3 \leq A_3$ so that $B_3 \leq \min(A_3, B_1)$.

Case 3. $A_1 \leq B_2, A_2, B_1$ and $B_3 \leq A_2, A_3, B_2$.
Then $A_1 \leq B_1$ and $B_3 \leq A_3$ so that $\min(A_1, B_3) \leq \min(A_3, B_1)$.

Case 4. $B_2 \leq A_1, A_2, B_1$ and $A_2 \leq B_3, A_3, B_2$.
Then $A_2 = B_2$ and we have item 2 indifferent to item 1 and item 3. In this case, item 1 may or may not precede item 3, but there is no contradiction to transitivity as long as we order item 1 and item 3 first and then put item 2 anywhere.

If Relation (2) is used, there is an extremely simple, practical way of ordering the items in n steps.

Working Rule.

1. List the A's and B's in two vertical columns.

i	A_i	B_i
1	A_1	B_1
2	A_2	B_2
.	.	.
.	.	.
.	.	.
n	A_n	B_n

2. Scan all the time periods for the shortest one.
3. If it is for the first machine (i.e., an A_i), place the corresponding item first.
4. If it is for the second machine (i.e., a B_i), place the corresponding item last.
5. Cross off both times for that item.

6. Repeat the steps on the reduced set of $2n-2$ time intervals, etc. Thus we work from both ends toward the middle.

7. In case of ties, for the sake of definiteness order the item with the smallest subscript first. In case of a tie between A_i and B_i order the item according to the A.

To illustrate the method, the following somewhat extreme example is worked out. Consider

i	A_i	B_i
1	4	5
2	4	1
3	30	4
4	6	30
5	2	3

The rule gives an optimal sequence (5, 1, 4, 3, 2). The total delay time for this sequence is four units, and the total elapsed time is 47 units. If one reversed the order of the items, the total time would be 78 units, the worst value possible.

2. THREE-STAGE PRODUCTION SCHEDULE

For three different machines or stages (at most, one item at a time on each machine), the problem loses some of the nice structure of the two-stage case. The problem is formulated, however, and for the special cases where min $A_i \geq \max B_j$ or min $C_i \geq \max B_j$ the complete solution is found analogously to the two-stage problem.

Lemma 3. An optimal ordering can be reached if we assume the same ordering of the n items for each machine.

Proof. By Lemma 1 the orders on the first and third machines can be made the same as that of the second; i.e., the first two machines have the same orders and the last two machines have the same orders. Thus the lemma is proved.

For four or more stages, the optimal scheduling may call for a shift in ordering of the items. Consider two items going through four stages with times listed below:

i	A_i	B_i	C_i	D_i
1	3	3	3	3
2	3	1	1	3

It can be verified that the optimal scheduling here calls for a shift of ordering from the second to the third stage. Thus the general solution is apt to be very complicated.

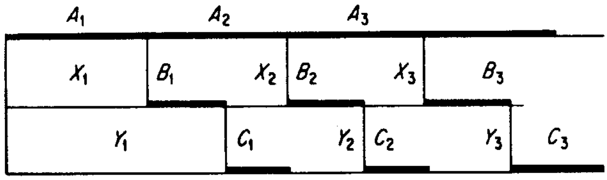
Notation.

Let A_i, B_i, X_i be defined as in the two-stage problem.

Let C_i = set-up time plus work time for the ith item on the third machine.

Let Y_i = the delay interval on the third machine immediately preceding the entry of the ith item onto the third machine.

Consider the time scales for each machine.



We have

$$Y_1 = X_1 + B_1 = A_1 + B_1$$

$$Y_n = \max\left(\sum_1^n B_i + \sum_1^n X_i - \sum_1^{n-1} C_i - \sum_1^{n-1} Y_i, 0\right)$$

so that

$$\sum_1^n Y_i = \max\left(\sum_1^n B_i - \sum_1^{n-1} C_i + \sum_1^n X_i, \sum_1^{n-1} Y_i\right)$$

$$= \max\left(\sum_1^n B_i - \sum_1^{n-1} C_i + \sum_1^n X_i, \right.$$

$$\left. \sum_1^{n-1} B_i - \sum_1^{n-2} C_i + \sum_1^{n-1} X_i, \ldots, B_1 + X_1\right).$$

Let

$$H_v = \sum_1^v B_i - \sum_1^{v-1} C_i, \qquad v = 1, 2, \ldots, n$$

and

$$K_u = \sum_1^u A_i - \sum_1^{u-1} B_i, \qquad u = 1, 2, \ldots, n, \text{ as before.}$$

Then

$$\sum_1^n Y_i = \max_{1 \leq u \leq v \leq n} (H_v + \max K_u) = \max_{1 \leq u \leq v \leq n} (H_v + K_u).$$

As before, we interchange the jth and $j+1$-st items. Then the H's and K's are unchanged except possibly those with subscripts j and $j+1$.

Now we compare

$$\max (H_{j+1} + K_u, 1 \leq u \leq j+1; H_j + K_u, 1 \leq u \leq j)$$

with $\max(H'_{j+1} + K'_u, 1 \leq u \leq j+1; H'_j + K'_u, 1 \leq u \leq j)$.

Notice that these terms no longer involve just the subscripts j and $j+1$, and thus the decision is not independent of what precedes the interchanged elements.

3. SPECIAL CASE WHERE MIN $A_i \geq$ MAX B_j

Here $\max_{u \leq v} K_u = K_v$, so that we now compare fewer terms. Our rule now states that the jth item precedes the $j+1$-st item if

$$\max(H_{j+1} + K_{j+1}, H_j + K_j) < \max(H'_{j+1} + K'_{j+1}, H'_j + K'_j). \quad (3)$$

In case of equality, we make the ordering of indifferent items consistent with the ordering given by the definite inequalities.

Then by subtracting

$$\sum_1^{j+1} A_i - \sum_1^{j-1} B_i + \sum_1^{j+1} B_i - \sum_1^{j-1} C_i$$

from two sides of (3), it becomes

$$\max(-B_j - C_j, -B_{j+1} - A_{j+1}) < \max(-B_{j+1} - C_{j+1}, -B_j - A_j)$$

or

$$\min(A_j + B_j, C_{j+1} + B_{j+1}) < \min(A_{j+1} + B_{j+1}, C_j + B_j). \quad (4)$$

Lemma 4. Relation (4) is transitive.

Proof. Proof is the same as for Lemma 2.

By the same arguments as before, we can reach an optimal sequence by successive interchanges of adjacent elements in any sequence following this rule. Thus we have

Theorem 2. If $\min A_i \geq \max B_i$, $1 \leq i \leq n$, then an optimal three-stage production schedule is given by the following rule:
Item i precedes item j if

$$\min(A_i + B_i, C_j + B_j) < \min(A_j + B_j, C_i + B_i).$$

If equality holds, the two items are indifferent and either is permissible, provided we order these items in a manner consistent with the orders given by the definite inequalities.

As in the two-stage case, there is a short working rule providing the optimal scheduling very quickly. Here A_i is replaced by $A_i + B_i$, and B_j is replaced by $B_j + C_j$.

Note that the same results hold if $\min C_i \geq \max B_j$.

Another special three-stage case is when the two-stage rule applied to the first two stages gives the same ordering as that for the last two stages. Then this ordering is the optimal for the three-stage case.

An equivalent statement of the three-stage problem is as follows: Notice that

$$\max_{u \leq v \leq n} \left(K_u + H_v + \sum_{i=1}^{n} C_i \right)$$

is the maximum sum of elements passed through on all "walks" in the time matrix from the upper left-hand corner to the lower right-hand corner, when steps are taken to the right or downward. The problem is to find a scheduling of items which minimizes this maximum walk.

This interpretation is useful in numerical work for three-stage problems but does not carry over to four or more stages.

As we noted previously, the optimal ordering is not always the same on each stage when there are more than three stages. As a practical working rule, however, one could assume the same order for each stage and then use the "maximum walk" interpretation to eliminate candidates for an optimal schedule.

Author's Note. Since this paper was written in 1953, there have been several extensions and generalizations by various authors, for example, S.M. Johnson, "Discussion: Sequencing in Jobs on Two Machines with Arbitrary Time Lags," *Management Science* 5, 3, (April, 1959).