

A Curiosity about Dudeney's Curious Numbers

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Nov. 6, 2009 (almost 1000 years later than Dudeney)

In "Amusements in Mathematics", Henry Ernest Dudeney [1] observes the following curiosity:

114. Curious Numbers

The number 48 has this peculiarity, that if you add 1 to it the result is a square number (49, the square of 7), and if you add 1 to its half, you also get a square number (25, the square of 5). Now, there is no limit to the numbers that have this peculiarity, and it is an interesting puzzle to find three more of them—the smallest possible numbers. What are they?

His solution:

The three smallest numbers, in addition to 48, are 1,680, 57,120, and 1,940,448. It will be found that 1,681 and 841, 57,121 and 28,561, 1,940,449 and 970,225, are respectively the squares of 41 and 29, 239 and 169, 1,393 and 985.

Essentially find x such that $x + 1$ and $x / 2 + 1$ are squares. For instance, the first number of this kind, 48, one can see $48 + 1$ is a square ($7 * 7$) and $48 / 2 + 1 = 25$ ($5 * 5$) is a square. He goes on to give the following sequence of numbers of this kind:

48, 1680, 57120, 1940448, ..

Of course this was back in 1900's when the computers were not readily available. With the availability of computers at our finger tips, we can do better and extend the sequence to:

48, 1680, 57120, 1940448, 65918160, 2239277040, 76069501248, 2584123765440, 87784138523760, 2982076586042448, 101302819786919520, 3441313796169221280, ..

What is highly curious about Dudeney's curious numbers is that there is a pattern to it not immediately obvious to the naked eye. If we take the ratios between consecutive numbers (like $1680 / 48$, $57120 / 1680$, ..), we get:

35, 34,
33.9714285714285680, 33.9705882352941160, 33.9705634987384360,
33.9705627705627720, 33.9705627491272820, 33.9705627484962780,
33.9705627484777040, 33.9705627484771570, 33.9705627484771430, ..

Thus we see the ratio converging to $33.9705627 \dots$ which happens to be the same as $(17 + 12\sqrt{2})$. One can see from Sloane's A01108 sequence we have:

*a(n)-th triangular number is a square: $a(n+1) = 6*a(n)-a(n-1)+2$, with $a(0) = 0$, $a(1) = 1$.*

0, 1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, 11309768, 65918161, 384199200, 2239277041, 13051463048, 76069501249, 443365544448, 2584123765441, 15061377048200, 87784138523761, 511643454094368,...

Our sequence shown above in green has the almost same numbers (one off) shown in red. In fact it can be seen that for 49 we have $\frac{1}{2} * 49 * (49 + 1) = 49 * 25$, a product of two squares. For numbers highlighted in red, the triangular numbers (i.e., $\frac{1}{2} * n * (n + 1)$) are a product of two odd squares and from these numbers we can get Dudeney's sequence. They have to be a product of two odd squares for Dudeney's sequence and that is why we have to take alternate terms.

Benoit Cloitre gives the following formula for Sloane's sequence A01108:

$a(n) = \text{floor}((1/4) * (3+2*\sqrt{2})^n)$ - Benoit Cloitre (benoit7848c(AT)orange.fr), Sep 04 2002

and since we are interested in the alternate terms, we can see our required ratio as:

$$(3 + 2 * \sqrt{2})^2 = (17 + 12 * \sqrt{2}) = 33.970562748477\dots$$

Sloane does list Dudeney's sequence as A008845 but with no connection to A01108

[1] Gutenberg Project. Dudeney, H.E., Amusements in Mathematics, 1917.