A Coin Problem or How Many Ways can Bill Gates Distribute his Fortune in Coins ?

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This is a simple high school problem (brought to the author's attention by Jimmy Wang) that appears to be computationally complex to solve but has solutions that can be written down (conjecturally) in a deterministic way. It can be stated as follows. Assuming American coinage of 1 (penny), 5 (nickel), 10 (dime) and 25 (quarter) cents, the problem is to determine how many ways a given quantity of money expressed in cents be changed using these four denominations. For instance five cents can be changed into two ways: one nickel or five pennies. Twenty cents can be changed into nine ways: two dimes, one dime + 2 nickels, one dime + one nickel + 5 pennies, one dime + 10 pennies, 4 nickels, 3 nickels + 5 pennies, 2 nickels + 10 pennies, one nickel + 15 pennies, and 20 pennies. This appears to be a NP problem as there seems to be no simple polynomially complex procedures exist to evaluate the number of ways for a given amount.

However the problem, known in the literature as a restricted partition problem, holds some deep underlying unfathomable structure (at least to the author) that makes it possible to compute the number of ways (conjecturally – what I mean by this will become clear later on) for arbitrarily large amounts. For instance one can say (conjecture) with reasonable certainty that Bill Gates' fortune at 40 billion dollars (or 4000 billion cents) can be distributed, if he wants to (most probably he does not want to), in 853333333347733333333404000000000001 ≈ 8.5E33 ways. There is also another amazing property about these numbers: if C(m) is the number of ways the amount m can be changed, then $C(m) / C(n) \approx (m/n)^3$ as m and n → ∞. As an example, we have C(9000000) = 97200729001590001, C(8000000) = 68267242668080001 and C(1000000) = 133342333510001. These are computed exactly using an enumerative algorithm and C(9000000) took 194918 seconds or 54 hours on a present day PC. One can note C(9000000) / C(1000000) = 728.95626 $\approx 729 = (9/1)^3$, C(9000000) / C(8000000) =1.423827 $\approx (9/8)^3$ = 1.423828, and C(9000000) / C(7000000) = 2.125360 $\approx (9/7)^3$ = 2.125364 thus showing that for larger m and n $C(m) / C(n) \to (m/n)^3$. Partition problems have been studied extensively (especially after Ramanujan-Hardy and Rademacher established almost exact formulas for the number of partitions [1]) and these kinds of results seem to be unknown.

Let us delineate the process by which we arrived at these startling facts. An enumerative procedure was written (in C/C++) to compute the number of ways a given quantity can be distributed or changed and the following table gives the values computed for a given quantity.

Table 1. Number of ways C(Q) to distribute Q using coinage of {1, 5, 10, 25}

Q	C(Q)
100	242
1000	142511
10000	134235101
100000	133423351001
1000000	133342333510001
1<6>0	**1<3>342<3>351<3>01**
1<n>0	**1<n-3>342<n-3>351<n-3>01**
200	1463
2000	1103021
20000	1070270201
200000	1067026702001
2000000	1066702667020001
2<6>0	**10<2>6702<2>6702<3>01**
2<n>0	**10<n-4>6702<n-4>6702<n-3>01**
300	4464
3000	3681531
30000	3608105301
300000	3600810053001
3000000	3600081000530001
3<6>0	**36<3>081<3>053<3>01**
3<n>0	**36<n-3>081<n-3>053<n-3>01**
400	10045
4000	8678041
40000	8547740401
400000	8534773404001
4000000	8533477334040001
4<6>0	**85<2>3477<2>3404<3>01**
4<n>0	**85<n-4>3477<n-4>3404<n-3>01**
4<12>0 (Bill Gates' Worth)	**85<8>3477<8>3404<9>01**
5<n>0	**1<n-2>6891<n-4>6755<n-3>01**
6<n>0	**288<n-4>0324<n-4>0106<n-3>01**
7<n>0	**457<n-4>3774<n-4>3457<n-3>01**
8<n>0	**682<n-5>67242<n-4>6808<n-3>01**
9<n>0	**972<n-4>0729<n-4>0159<n-3>01**

In the table we follow the notation whereby <m>x indicates the digits xxx..x where there are m x's. Thus 1<3>5 stands for 1555 and 1<3>546<2>01 stands for 155546001.

We note a remarkable pattern emerging for larger values of Q. For 100000, we have 133423351001 and 1000000, we have 133342333510001. Observing the patterns for

10000, 100000 and 1000000, we boldly conclude **C(1<n>0)** = **1<n-3>342<n-3>351<n-3>01**. Note this is conjectural and we would like to go beyond 9000000 (nine million) to verify the pattern is persisting. Unfortunately the computational resources available at this time prevents such a task but based on the emerging patterns, we make these conjectures. Also the cubic conjecture, i. e., $C(m) / C(n) \rightarrow (m/n)^3$ is highly interesting and we have no clue whatsoever how to prove this.

One may question the utility of the whole thing. Only response I have is that there are things unknown (or not revealed to us) at this time and it is human nature to explore things unknown to get a better understanding of things we understand and things we do not understand.

Available at www.rspq.org/pubs - This is an internet publication and will not appear in any journals.

Reference:

[1] Wilf, http://www.math.upenn.edu/~wilf/PIMS/PIMSLectures.pdf.